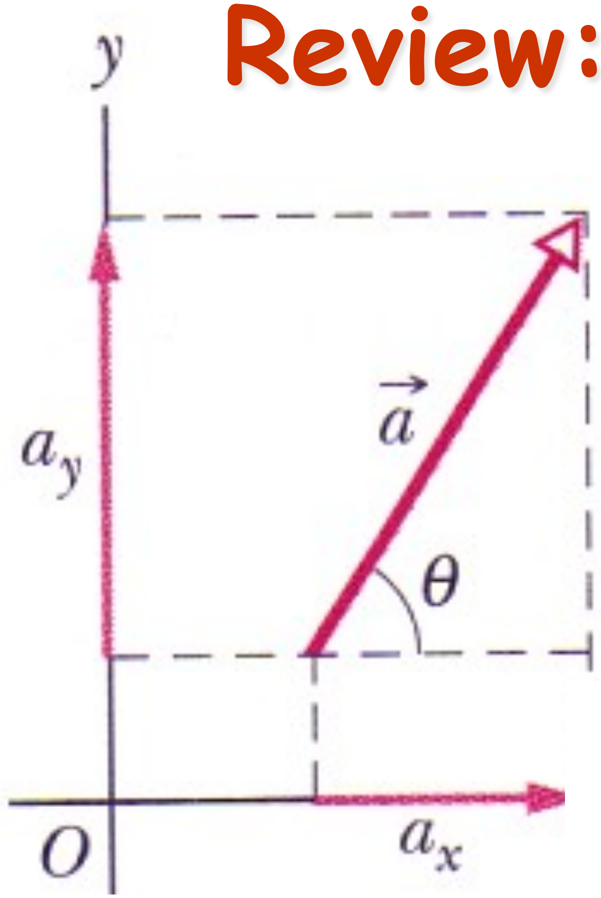
Chapter 3: 2D Kinematics
Tuesday January 20th

- Chapter 3: Vectors
  - Review: Properties of vectors
  - Review: Unit vectors
  - Position and displacement
  - Velocity and acceleration vectors
  - Relative motion
- Constant acceleration in 2D and 3D
  - Projectile motion
  - Demonstrations and examples

Reading: up to page 41 in the text book (Ch. 3)

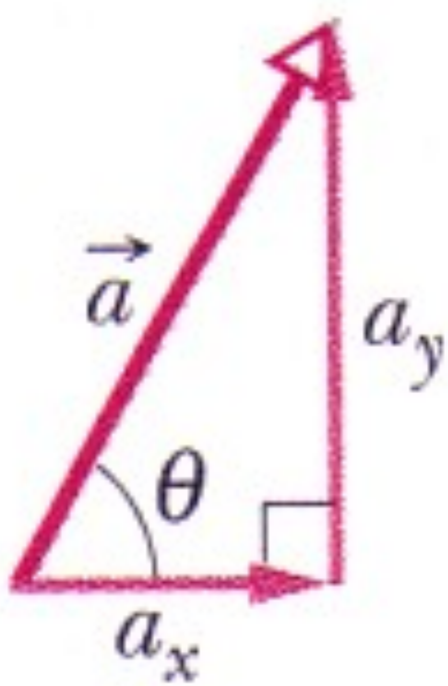
Review: Components of vectors



Resolving vector components

$$a_x = a\cos\theta$$

$$a_y = a\sin\theta$$

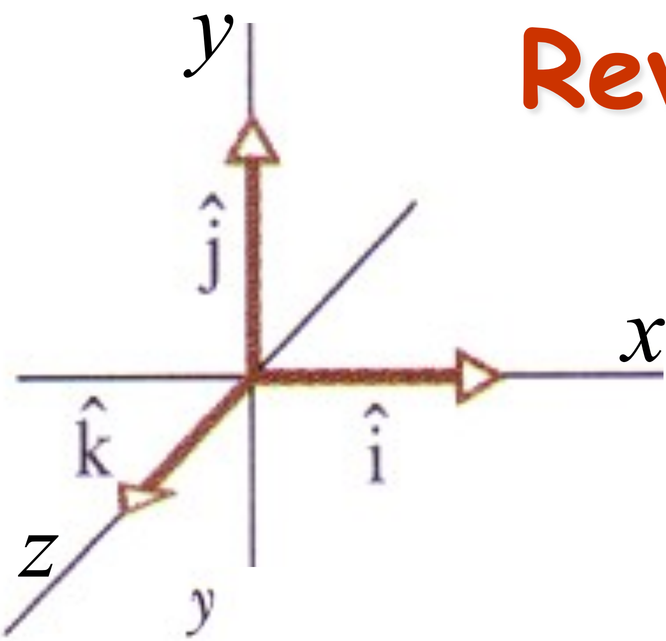


The inverse process

$$a = \sqrt{a_x^2 + a_y^2}$$

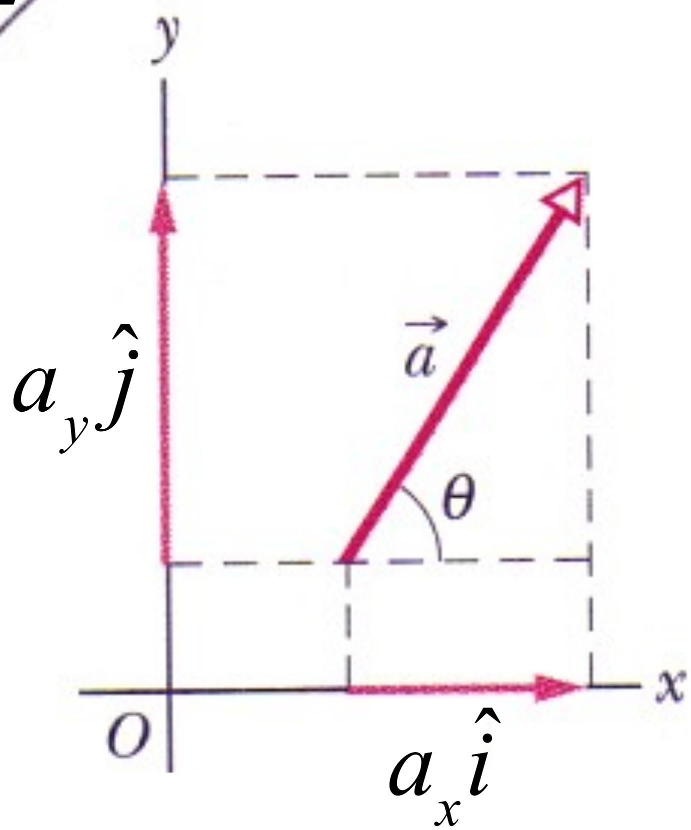
$$\tan\theta = \frac{a_y}{a_x}$$

Review: Unit vectors



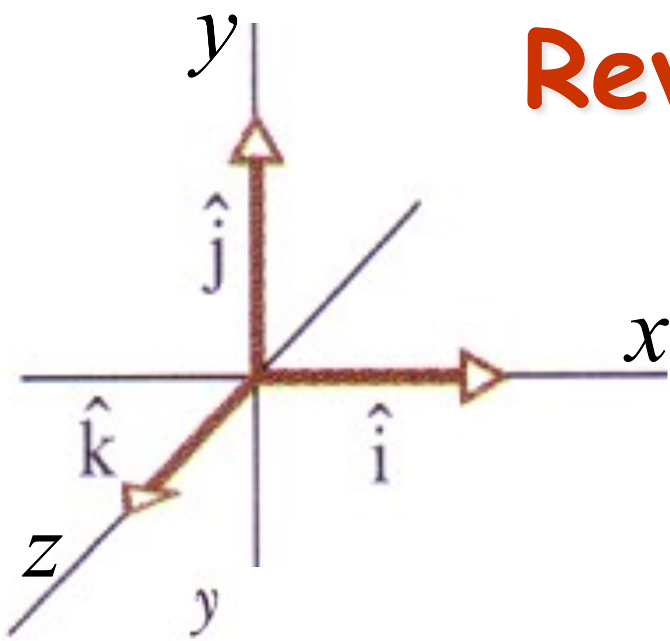
$\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors

They have length equal to unity (1), and point respectively along the x, y and z axes of a right handed Cartesian coordinate system.



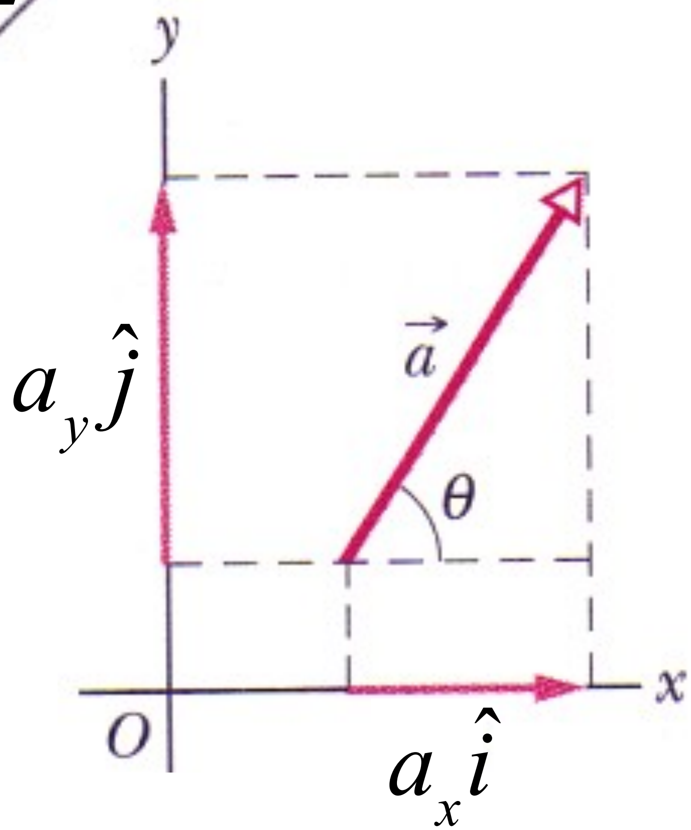
$$\vec{a} = a_x\hat{i} + a_y\hat{j}$$

Review: Unit vectors



$\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors

They have length equal to unity (1), and point respectively along the x, y and z axes of a <u>right handed Cartesian</u> coordinate system.



$$\vec{a} = a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j}$$

Note: θ is **<u>usually</u>** measured from x to y (in a right-handed sense around the z-axis)

Adding vectors by components

Consider two vectors:

$$\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

&

$$\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$

Then…

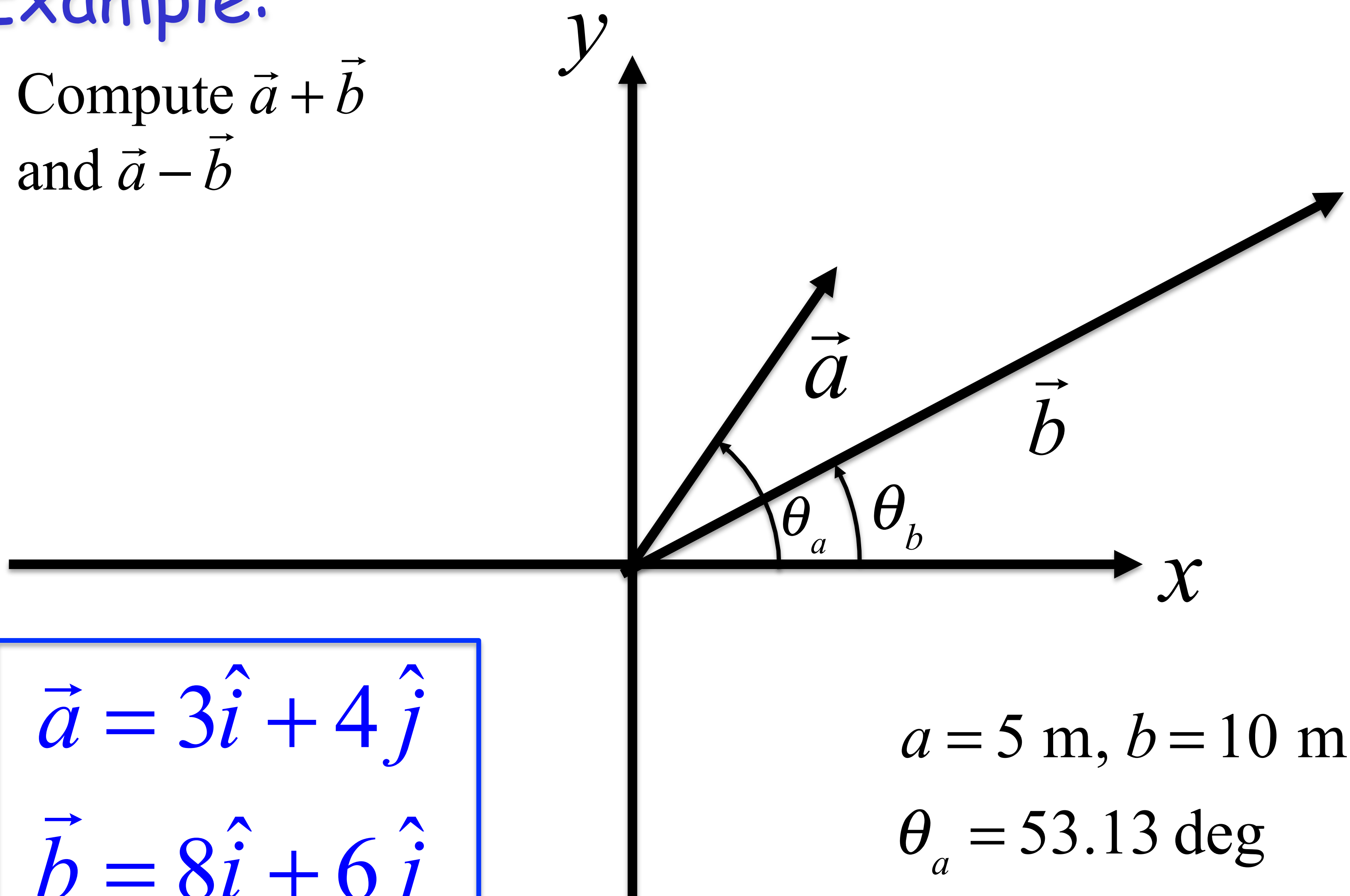
$$\vec{r}_2 - \vec{r}_1 = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

&

$$\vec{r}_1 + \vec{r}_2 = (x_2 + x_1)\hat{i} + (y_2 + y_1)\hat{j} + (z_2 + z_1)\hat{k}$$

Example:

Compute $\vec{a} + \vec{b}$
and $\vec{a} - \vec{b}$



$$\vec{a} = 3\hat{i} + 4\hat{j}$$

$$\vec{b} = 8\hat{i} + 6\hat{j}$$

$a = 5$ m, $b = 10$ m

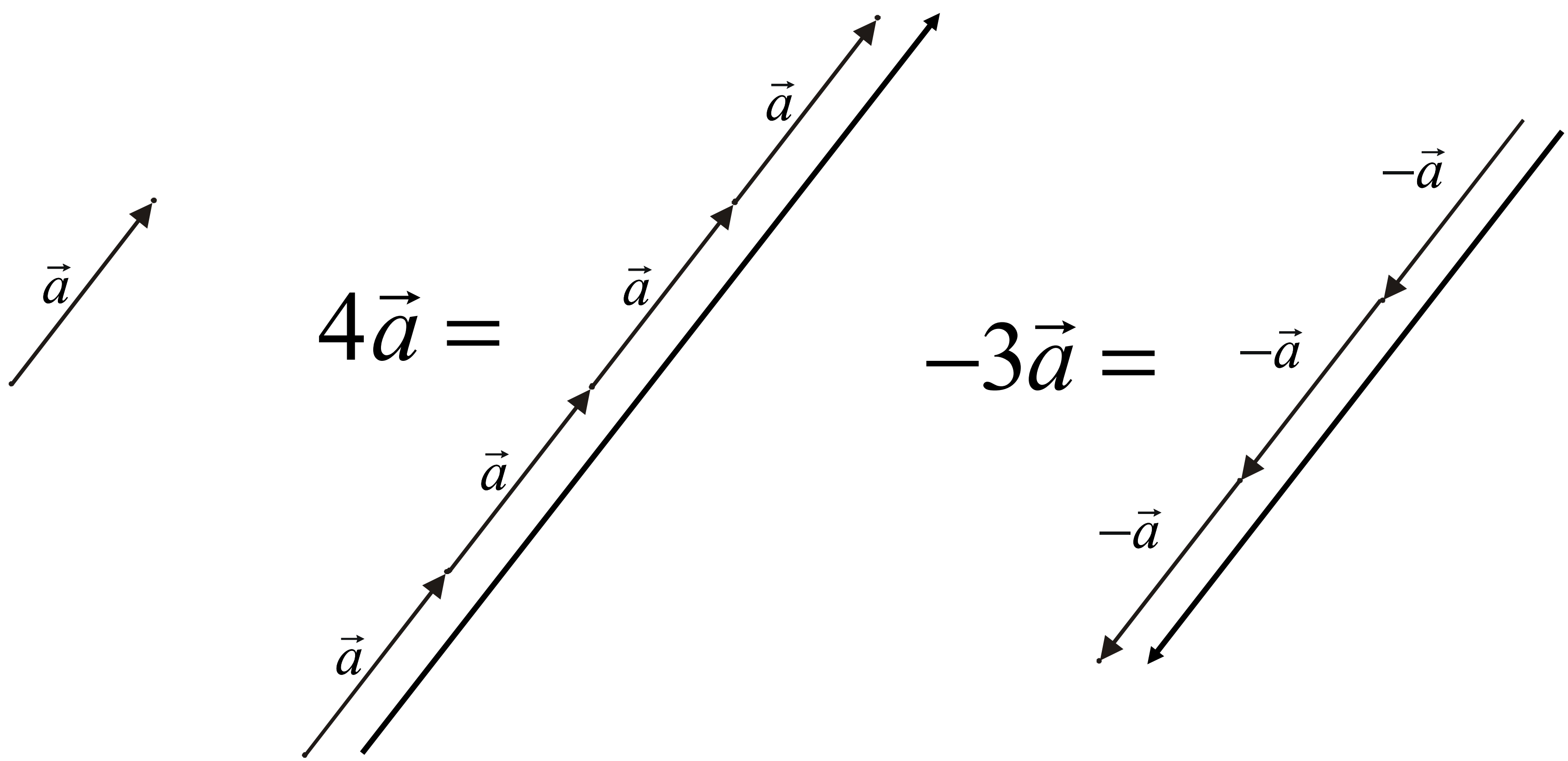
$\theta_a = 53.13$ deg

$\theta_b = 36.87$ deg

Multiplying vectors (very brief)

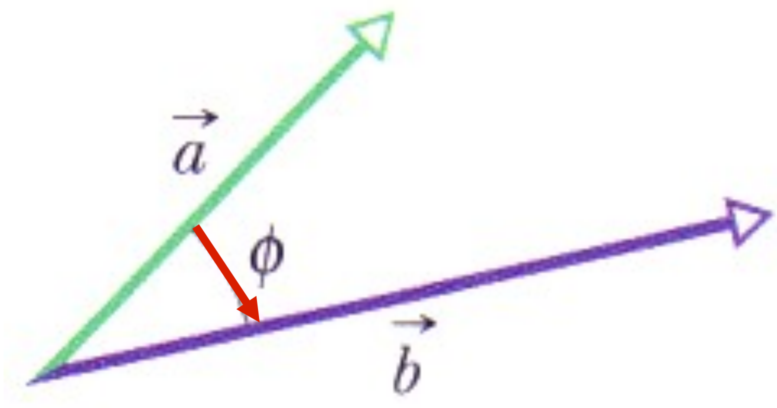
Multiplying a vector by a scalar:

- This operation simply changes the length of the vector, not the orientation.



The scalar product, or dot product

$$\vec{a}\cdot\vec{b} = ab\cos\phi$$



$$(a)(b\cos\phi) = (a\cos\phi)(b)$$

$$\cos\phi = \cos(-\phi)$$

$$\Rightarrow \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$

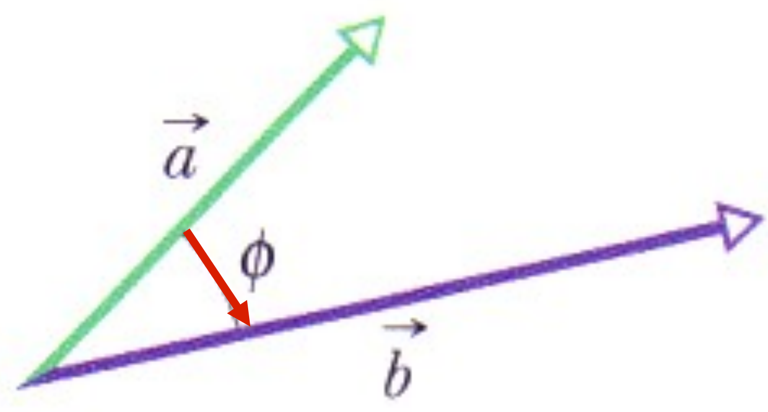
- The scalar product represents the product of the magnitude of one vector and the component of the second vector along the direction of the first

$$\text{If } \phi = 0^{\circ}, \text{ then } \vec{a}\cdot\vec{b} = ab$$

$$\text{If } \phi = 90^{\circ}, \text{ then } \vec{a}\cdot\vec{b} = 0$$

The scalar product, or dot product

$$\vec{a} \cdot \vec{b} = ab\cos\phi$$



$$(a)(b\cos\phi) = (a\cos\phi)(b)$$

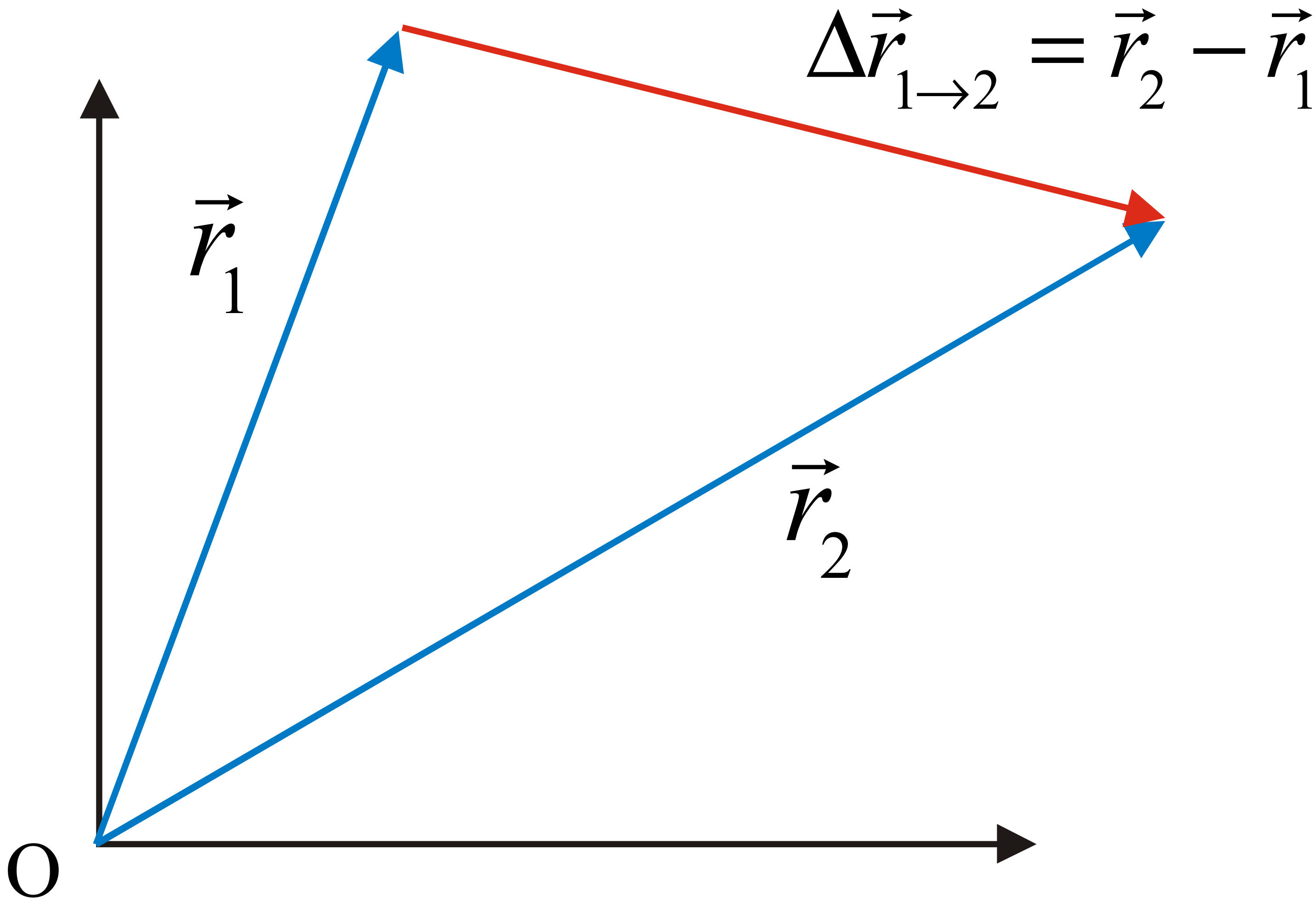
$$\cos\phi = \cos(-\phi)$$

$$\Rightarrow \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

- The scalar product becomes relevant in Chapter 6 when considering work and power.
- There is also a vector product, or cross product, which becomes relevant in Chapter 11. I save discussion of this until later in the semester - see also Appendix to this lecture.

Position and displacement

•Position vectors represent coordinates. The displacement between two position vectors is given as follows:



Average and instantaneous velocity

$$\text{average velocity} = \frac{\text{displacement}}{\text{time interval}} = \frac{\vec{r}_2 - \vec{r}_1}{\Delta t} = \frac{\Delta \vec{r}}{\Delta t}$$

Instantaneous velocity:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

Or:

$$v_x = \frac{dx}{dt}; \quad v_y = \frac{dy}{dt}; \quad v_z = \frac{dz}{dt}$$

Average and instantaneous acceleration

$$\vec{a}_{avg} = \frac{\text{change in velocity}}{\text{time interval}} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta \vec{v}}{\Delta t}$$

Instantaneous acceleration:

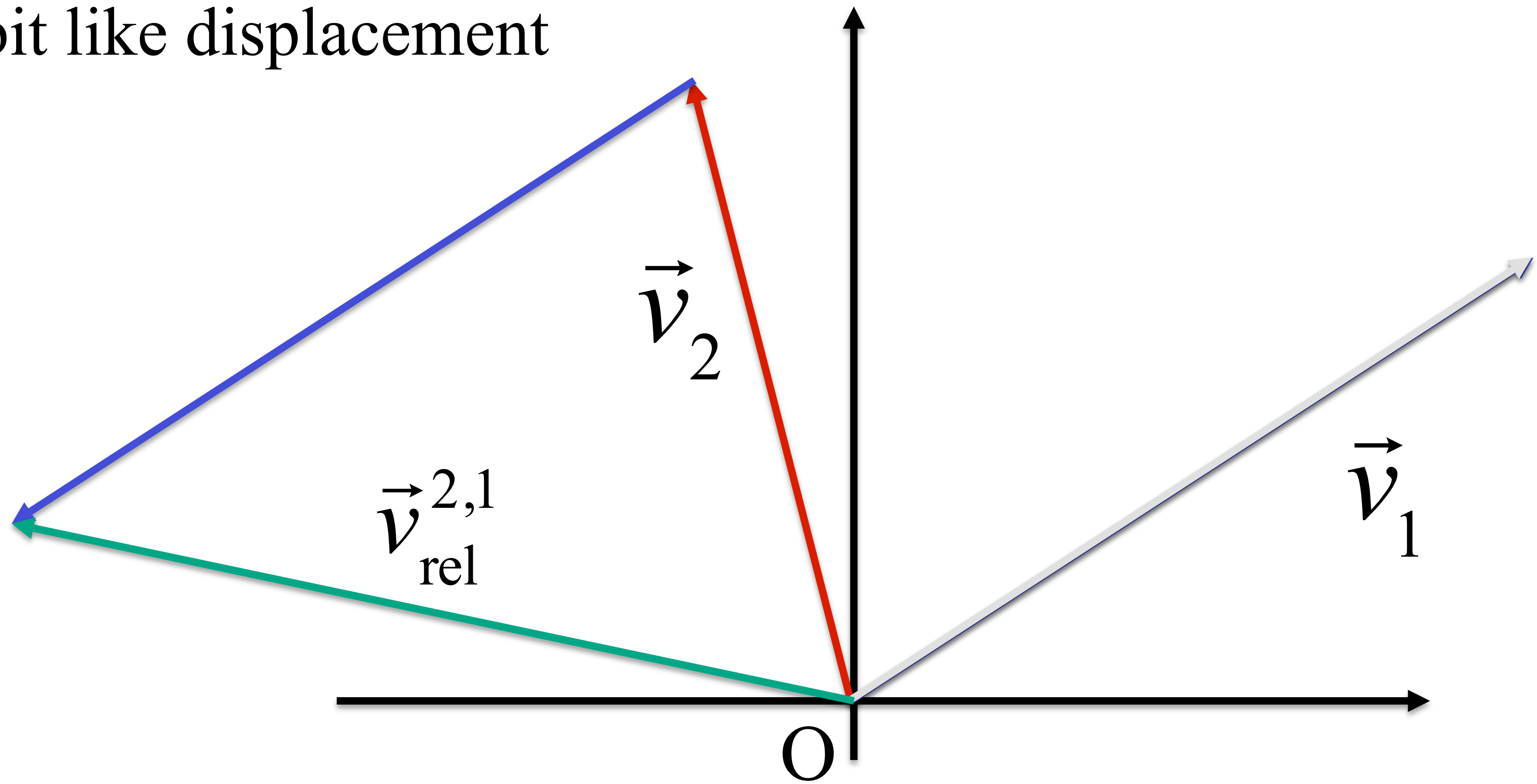
$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k}$$

Or:

$$a_x = \frac{dv_x}{dt}; \quad a_y = \frac{dv_y}{dt}; \quad a_z = \frac{dv_z}{dt}$$

Relative Velocity

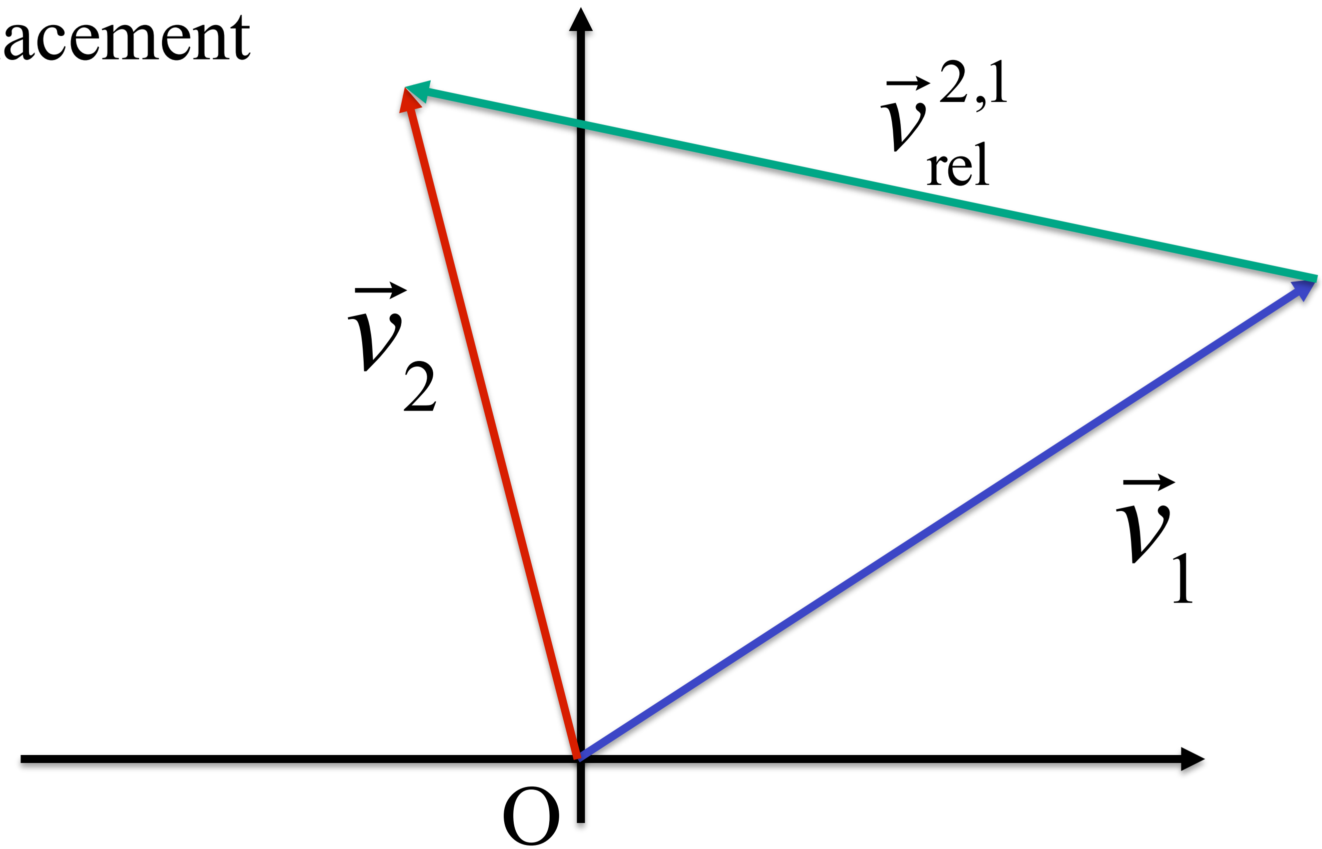
A bit like displacement



Velocity of 2 relative to 1: $\vec{v}_{\text{rel}}^{\,2,1} = \vec{v}_2 - \vec{v}_1$

Relative Velocity

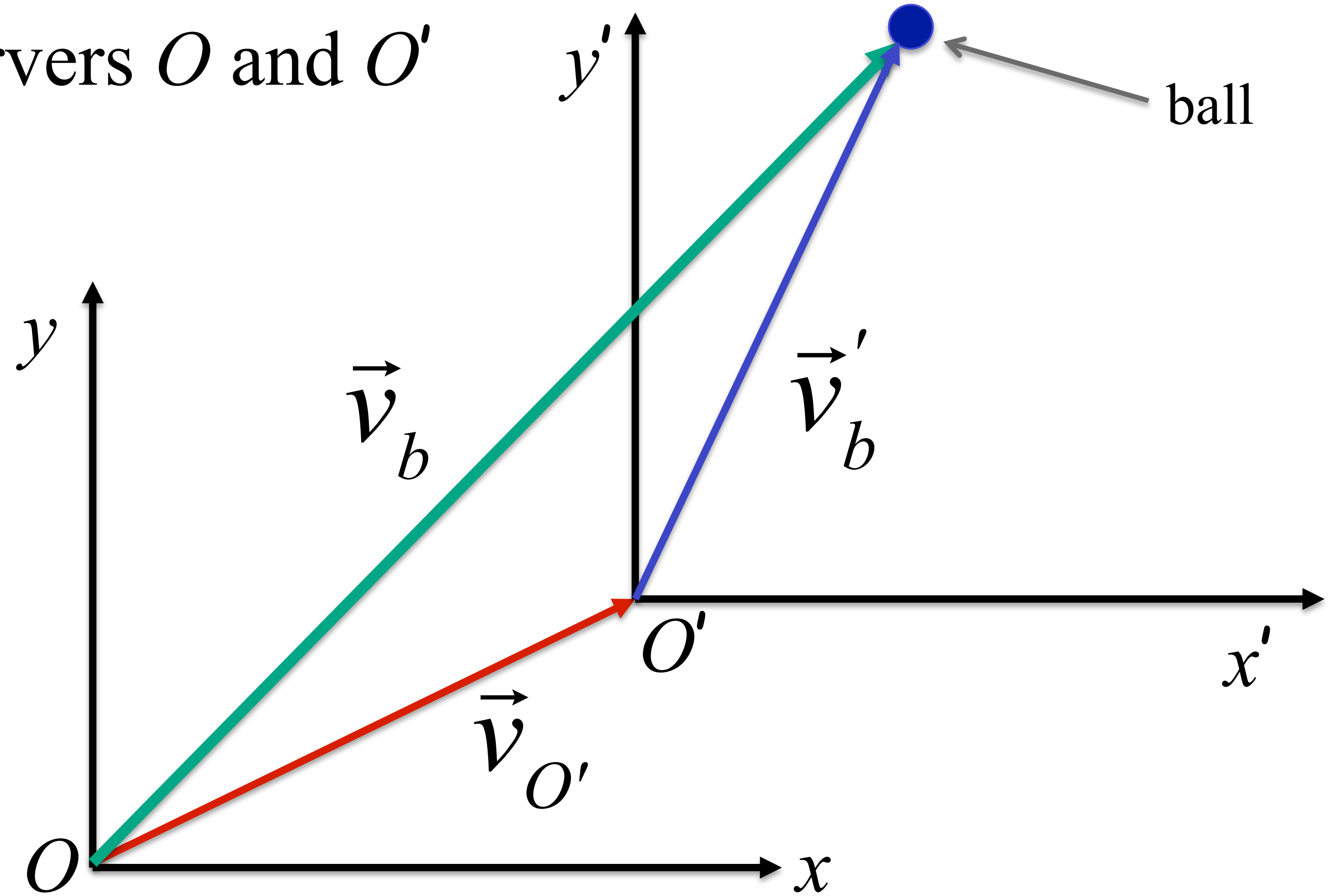
A bit like displacement



Velocity of 2 relative to 1: $\vec{v}_{rel}^{2,1} = \vec{v}_2 - \vec{v}_1$

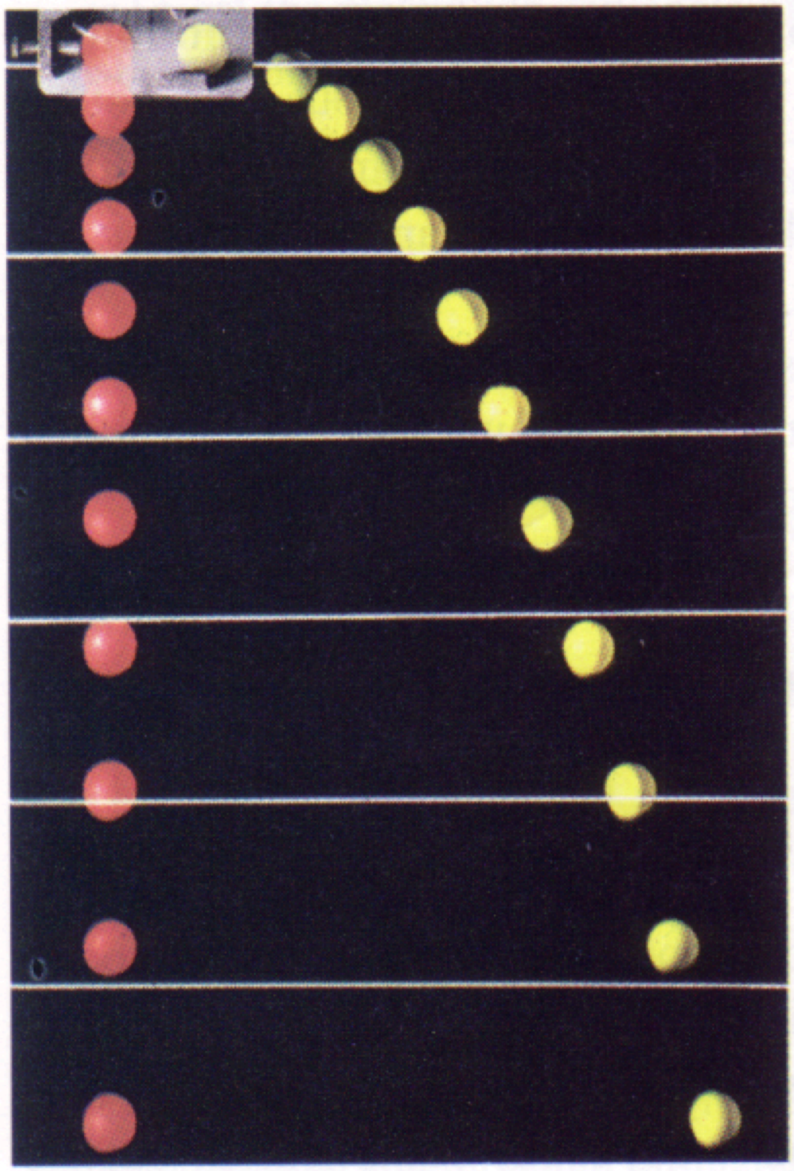
Relative Motion in Different Reference Frames

Two observers O and O'



Velocity of ball observed in O: $\vec{v}_b = \vec{v}_b' + \vec{v}_{O'}$

Projectile motion



- This series of photographic images illustrates the fact that vertical motion is unaffected by horizontal motion, i.e., the two balls accelerate downwards at the same constant rate, irrespective of their horizontal component of motion.
- In all of the projectile motion problems that we will consider, we shall assume that the only acceleration is due to gravity ($a = -g$) which acts in the $-y$ direction.

Displacement

- The equations of motion that we introduced in chapter 2 apply equally well in two and three-dimensional motion.
- All we need to do is to break the motion up into components, and treat each component independently.
- Displacement:

$$\Delta\vec{r}_{1\to 2} = \left(x_2 - x_1\right)\hat{i} + \left(y_2 - y_1\right)\hat{j} + \left(z_2 - z_1\right)\hat{k}$$

- Then we expect to have a different set of kinematic equations for each vector component.

Equations of motion for constant acceleration

Equation number	Equation	Missing quantity
3.8	$\vec{v} = \vec{v}_0 + \vec{a}t$	$(\vec{r} - \vec{r}_0)$
3.9	$\vec{r} - \vec{r}_0 = \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$	$\vec{v}$
	$v^2 = v_0^2 + 2\vec{a}\cdot(\vec{r} - \vec{r}_0)$	t
	$(\vec{r} - \vec{r}_0) = \frac{1}{2}(\vec{v}_0 + \vec{v})t$	$\vec{a}$
	$(\vec{r} - \vec{r}_0) = \vec{v}t - \frac{1}{2}\vec{a}t^2$	$\vec{v}_0$

Important: equations apply ONLY if acceleration is constant.

Equations of motion for constant acceleration

These equations work the same in any direction, e.g., along x, y or z.

Equation number	Equation	Missing quantity
2.7	$v_x = v_{0x} + a_x t$	$x - x_0$
2.10	$x - x_0 = v_{0x} t + \frac{1}{2} a_x t^2$	v_x
2.11	$v_x^2 = v_{0x}^2 + 2a_x (x - x_0)$	t
2.9	$x - x_0 = \frac{1}{2}(v_{0x} + v_x)t$	a_x
	$x - x_0 = v_x t - \frac{1}{2} a_x t^2$	v_0

Important: equations apply ONLY if acceleration is constant.

Equations of motion for constant acceleration

These equations work the same in any direction, e.g., along x, y or z.

Equation number	Equation	Missing quantity
2.7	$v_y = v_{0y} + a_y t$	$y - y_0$
2.10	$y - y_0 = v_{0y} t + \frac{1}{2} a_y t^2$	v_y
2.11	$v_y^2 = v_{0y}^2 + 2a_y (y - y_0)$	t
2.9	$y - y_0 = \frac{1}{2}(v_{0y} + v_y)t$	a_y
	$y - y_0 = v_y t - \frac{1}{2} a_y t^2$	v_{0y}

Important: equations apply ONLY if acceleration is constant.

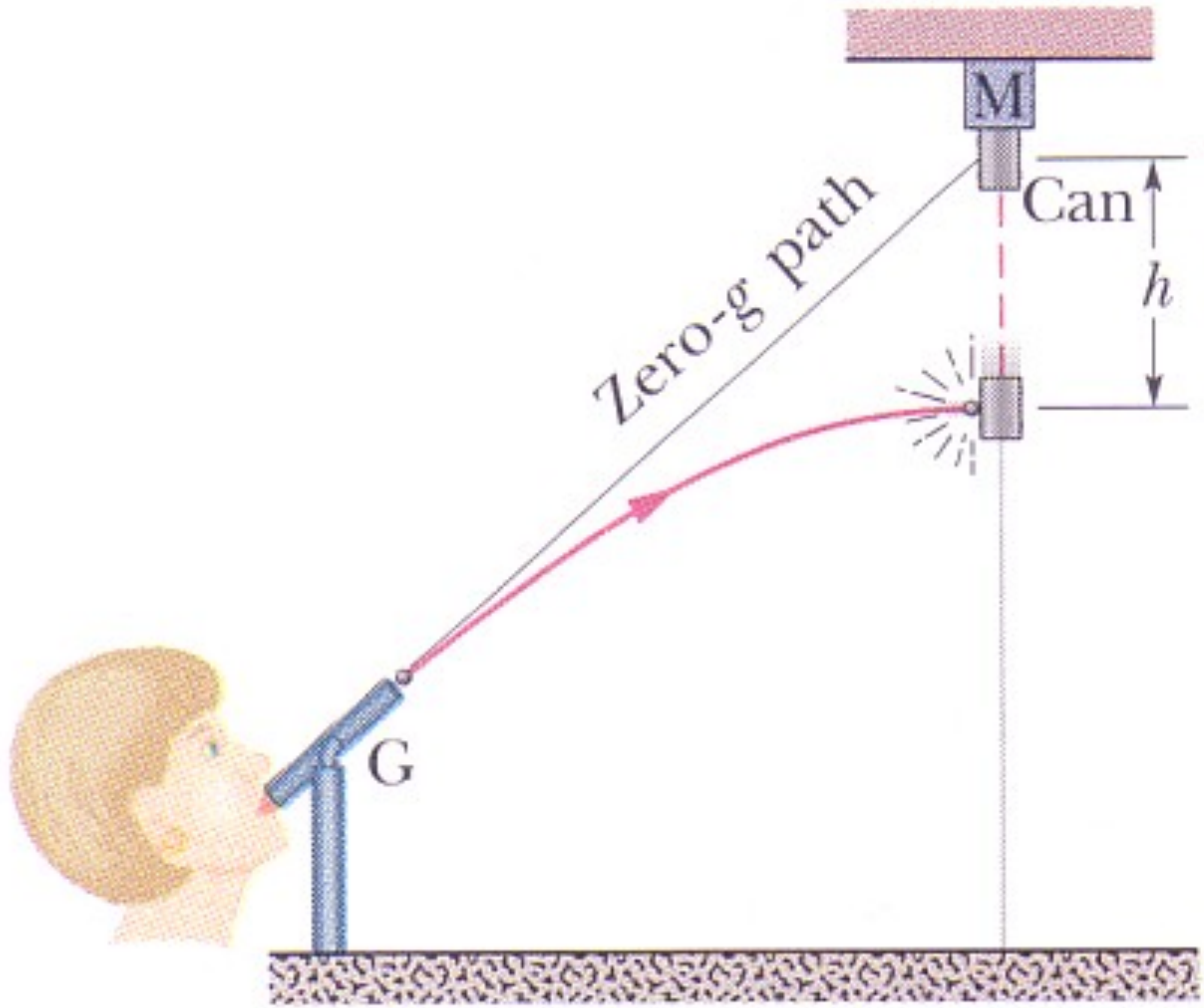
Equations of motion for constant acceleration

Special case of free-fall under gravity, $a_y = -g$.

$g = 9.81\ \mathrm{m/s^2}$ here at the surface of the earth.

Equation number	Equation	Missing quantity
	$v_y = v_{0y} - gt$	$y - y_0$
	$y - y_0 = v_{0y}t - \frac{1}{2}gt^2$	v_y
	$v_y^2 = v_{0y}^2 - 2g(y - y_0)$	t
	$y - y_0 = \frac{1}{2}(v_{0y} + v_y)t$	a_y
	$y - y_0 = v_y t + \frac{1}{2}gt^2$	v_{0y}

Demonstration



Appendices

The scalar product in component form

$$\vec{a}\cdot\vec{b}=\left(a_x\hat{\mathrm{i}}+a_y\hat{\mathrm{j}}+a_z\hat{\mathrm{k}}\right)\cdot\left(b_x\hat{\mathrm{i}}+b_y\hat{\mathrm{j}}+b_z\hat{\mathrm{k}}\right)$$

$$\vec{a}\cdot\vec{b}=a_xb_x+a_yb_y+a_zb_z$$

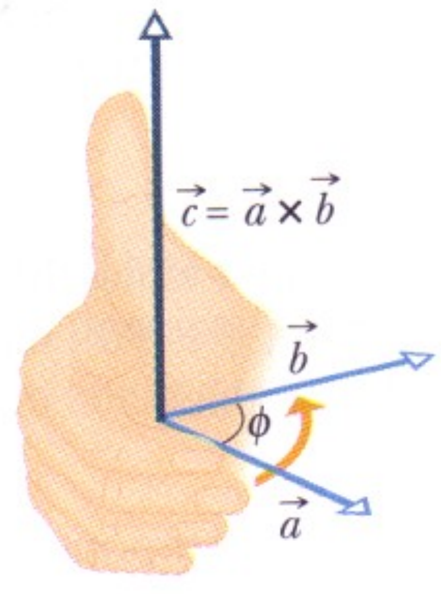
Because:

$$\hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}=1$$

$$\hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{i}=0$$

This is the property of orthogonality

The vector product, or cross product



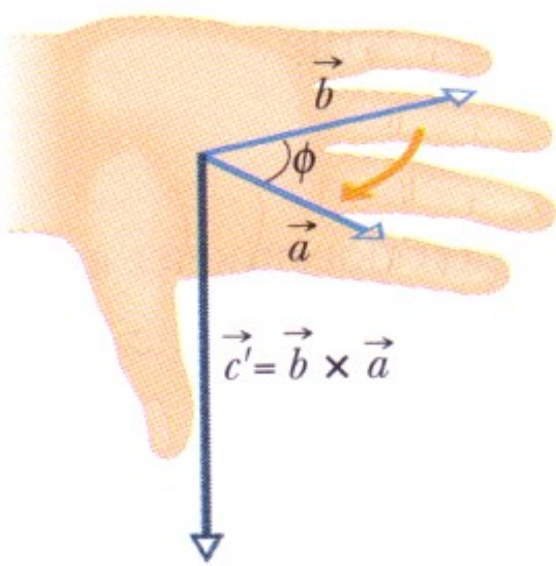
(a)

$$\vec{a} \times \vec{b} = \vec{c}, \text{ where } c = ab\sin\phi$$

$$\vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)$$

Direction of $\vec{c} \perp$ to both $\vec{a}$ and $\vec{b}$

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

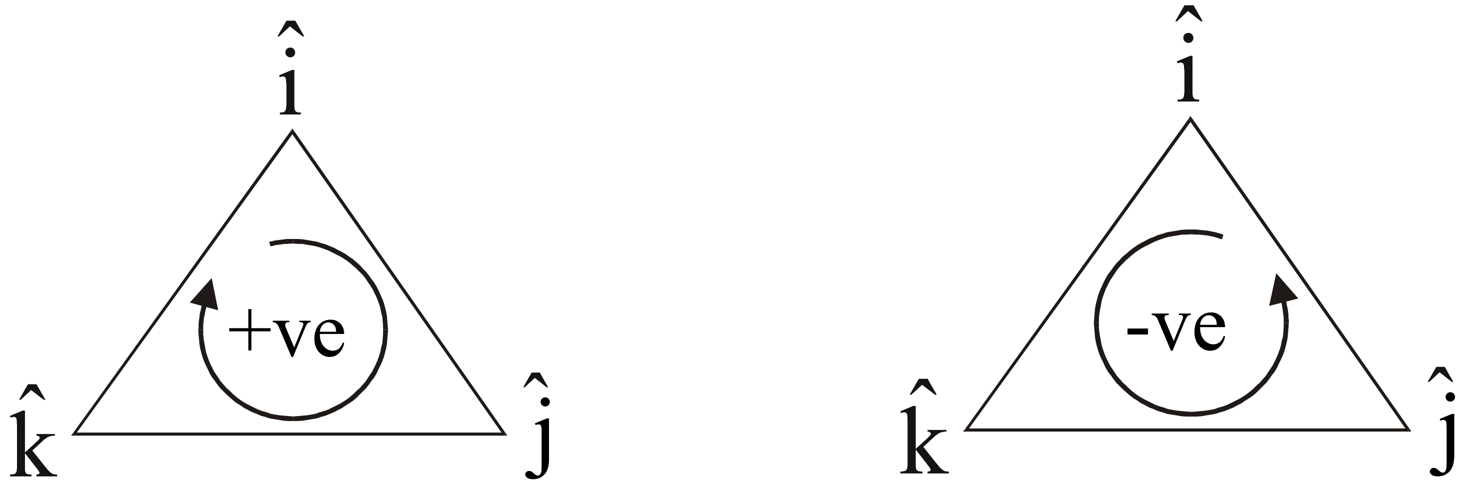


$$\hat{i} \times \hat{j} = \hat{k} \qquad \hat{j} \times \hat{i} = -\hat{k}$$

$$\hat{j} \times \hat{k} = \hat{i} \qquad \hat{k} \times \hat{j} = -\hat{i}$$

$$\hat{k} \times \hat{i} = \hat{j} \qquad \hat{i} \times \hat{k} = -\hat{j}$$

$$\vec{a}\times\vec{b}=\left(a_x\hat{i}+a_y\hat{j}+a_z\hat{k}\right)\times\left(b_x\hat{i}+b_y\hat{j}+b_z\hat{k}\right)$$



$$a_x\hat{i}\times b_y\hat{j}=a_xb_y\left(\hat{i}\times\hat{j}\right)=a_xb_y\hat{k}$$

$$\vec{a}\times\vec{b}=\left(a_yb_z-b_ya_z\right)\hat{i}+\left(a_zb_x-b_za_x\right)\hat{j}+\left(a_xb_y-a_yb_x\right)\hat{k}$$